• Last time we showed one direction of the following equivalences:

**Theorem**

*The following are equivalent:*

1. $\omega_1^{\mathcal{L}[x]} < \omega_1$.
2. $\text{PSP}(\Sigma^1_2(x))$.
3. $\text{PSP}(\Pi^1_1(x))$.

• Namely, (1) implies (2), and (2) implies (3).
• To get (3) implies (1), we need concepts related to prewellorders,
• Basically, a way to talk about the complexity of functions $f : X \to \text{Ord}$ for $X \subseteq \mathcal{N}$.
• This motivates the notion of a *scale* as well as bounds on the lengths of prewellorders.
Adequate Pointclasses

Because we’ll be talking a bit in the abstract, it helps to talk about properties common to the bold and lightface pointclasses:

**Definition**

A pointclass $\Gamma \subseteq \mathcal{P}(\mathcal{N})$ is *adequate* iff

- $\Gamma$ contains all computable relations;
- $\Gamma$ is closed under computable substitution/preimages;
- $\Gamma$ is closed under finite unions and intersections; and
- $\Gamma$ is closed under bounded quantification (over $\omega$).

- So clearly all borel, arithmetical, projective, and analytical pointclasses are adequate.
- Basically, simple operations don’t change complexity.
Defining Prewellorders

Definition

A prewellorder is a relation $\leq$ that is transitive, total, and well-founded.

- This is a prewellorder in the following sense.

Result

Let $\leq$ be a prewellorder on $X$. Define the equivalence relation $x \approx y$ iff $x \leq y \leq x$. Therefore $\leq/\approx$ is a well-order on $\{[x]_\approx : x \in X\}$.

- So what do prewellorders look like?
- Basically just well-orders with clusters of loops, but which don’t fundamentally change the order.
- So how do we get examples of prewellorders?
**Definition**

A **prewellorder** is a relation $\leq$ that is transitive, total, and well-founded.

**Definition**

A **norm** on a set $X$ is a function $\varphi : X \to \text{Ord}$.

**Result (19 C • 5)**

*Every prewellorder has a norm (its rank function). Moreover, every norm $\varphi : X \to \text{Ord}$ gives a prewellorder $x \leq y$ iff $\varphi(x) \leq \varphi(y)$.*

- Note that the norm associated with a prewellorder isn’t unique.
- For example, $\varphi = \{\langle 0, 0 \rangle, \langle 1, 1 \rangle\}$ and $\varphi' = \{\langle 0, 0 \rangle, \langle 1, 4 \rangle\}$ give the same prewellorder.
Defining Prewellorders

Result (19 C • 5)

Every prewellorder has a norm (its rank function). Moreover, every norm \( \varphi : X \to \text{Ord} \) gives a prewellorder \( x \leq y \iff \varphi(x) \leq \varphi(y) \).

- Also note that this gives lots of trivial prewellorders: any constant function \( \varphi : X \to \{\alpha\} \).
- On the other hand, assuming \( \varphi \) is injective, we get a well-order.
- Any set has lots of prewellorders, even in ZF.
- But we are interested in prewellorders with definability restrictions.
**Definition**

Let $\Gamma \subseteq \mathcal{P}(\mathcal{N})$ be a pointclass. $X \subseteq \mathcal{N}$ has a $\Gamma$-norm iff there’s a norm $\varphi : X \to \text{Ord}$ such that $\leq_\varphi$ and $<_\varphi$ are both in $\Gamma$, defined by

\[
\begin{align*}
  x \leq_\varphi y & \quad \text{iff} \quad x \in X \land (y \in X \rightarrow \varphi(x) \leq \varphi(y)) \\
  x <_\varphi y & \quad \text{iff} \quad x \in X \land (y \in Y \rightarrow \varphi(x) < \varphi(y))
\end{align*}
\]

- Let’s consider the complexity here: $x \leq_\varphi x$ iff $x \in X$ so having a $\Gamma$-norm implies $X \in \Gamma$ (if $\Gamma$ is adequate).
- We know the constant 0 function $\varphi = \text{const}_0$ is a norm on every set, but it may not have the best complexity:
  - $\text{const}_0(x) < \text{const}_0(y)$ is always false so that $x <_\varphi y$ iff $x \in X$ and $y \notin X$.
  - This requires complexity $\Gamma \land \neg \Gamma$.

**Corollary**

If $\Delta \subseteq \mathcal{P}(\mathcal{N})$ is adequate with $\neg \Delta = \Delta$, then every $X \in \Delta$ has a $\Delta$-norm. E.g. each $\Delta^1_n$ has this.
Definition

$\Gamma \subseteq \mathcal{P}(\mathcal{N})$ has the \textit{prewellordering property}, $\text{PWO}(\Gamma)$, iff every $X \in \Gamma$ has a $\Gamma$-norm.

Corollary

- $\text{PWO}(\Delta^0_\alpha)$ and $\text{PWO}(\Delta^1_n)$ for every $\alpha < \omega_1$, $n < \omega$ (and their \textit{boldface} counterparts).
- $\text{PWO}(\Sigma^1_0)$ (and $\text{PWO}(\Sigma^0_1)$)

Proof.

- $A \in \Sigma^1_0 = \Sigma^0_1$ is $\bigcup_{n \in \omega} \mathcal{N}_f(n)$ for some computable $f$.
- Define $\varphi(x)$ as the least $n \in \omega$ with $x \in \mathcal{N}_f(n)$.
- This is a $\Sigma^1_0$-norm: each cone is $\Delta^0_1$

\[
x \preceq_{\varphi} y \text{ iff } x \in A \land \exists n \in \omega \, (x \in \mathcal{N}_f(n) \land \forall m < n \, (y \notin \mathcal{N}_f(m)))
\]

\[
x \preceq_{\varphi} y \text{ iff } x \in A \land \exists n \in \omega \, (x \in \mathcal{N}_f(n) \land \forall m \leq n \, (y \notin \mathcal{N}_f(m)))
\]
The Prewellordering Property

**Definition**

\[ \Gamma \subseteq \mathcal{P}(\mathcal{N}) \] has the *prewellordering property*, \( \text{PWO}(\Gamma) \), iff every \( X \in \Gamma \) has a \( \Gamma \)-norm.

A much harder pointclass to show PWO for is \( \Pi_1^1 \).

**Theorem**

\( \text{PWO}(\Pi_1^1) \)

To prove this, recall the following nice properties for \( \Pi_1^1 \)-sets.

**Lemma**

*Every* \( X \in \Pi_1^1 \) *is the computable preimage of WO.*

This is quite nice for us, because we already showed that WO has a norm on it: \( x \mapsto \|x\| \) where \( \|x\| \) is the length of the well-order coded by \( x \in \text{WO} \).

**Lemma (19 B • 8 and 19 C • 8)**

\( \psi(x) = \|x\| \) *is a* \( \Pi_1^1 \)-norm on WO.*
**The Prewellordering Property**

**Lemma**

*Every* $X \in \Pi^1_1$ *is the computable preimage of* $WO$, *and* $\psi(x) = \|x\|$ *is a* $\Pi^1_1$-*norm on* $WO$.

**Theorem**

$PWO(\Pi^1_1)$

**Proof.**

As a result, we can define a $\Pi^1_1$-norm for $X \in \Pi^1_1$ as follows:

- Let $X = f^{-1}"WO$ for $f : \mathcal{N} \rightarrow \mathcal{N}$ computable.
- Define $\varphi = \|f \upharpoonright X\|$. We get that

$$x \leq_\varphi y \iff f(x) \leq_\psi f(y)$$

$$\iff f(x) \in WO \land (f(y) \in WO \rightarrow \|f(x)\| \leq \|f(y)\|)$$

$$\iff x \in X \land (y \in X \rightarrow \varphi(x) \leq \varphi(y)).$$
The Prewellordering Property

This also lifts to $\Sigma_2^1$.

**Corollary**

$$\text{PWO}(\Sigma_2^1) \text{ (and PWO}(\Sigma_2^1))$$

The idea is actually quite easy and nice, having the following form.

**Theorem**

*If $\Gamma$ is adequate and $X$ has a $\Gamma$-norm, then $\exists^N X$ has a $\exists^N \forall^N \Gamma$-norm. So $\text{PWO}(\Gamma)$ implies $\text{PWO}(\exists^N \forall^N \Gamma)$.*

**Proof.**

- Let $\varphi$ be a $\Gamma$-norm for $X$.
- Define $\psi : \exists^N X \to \text{Ord}$ just by
  $$\psi(x) = \min\{\varphi(x, y) : \langle x, y \rangle \in X\}.$$  
- This is a $\exists^N \forall^N \Gamma$-norm:
  $$x \leq_{\psi} y \quad \text{iff} \quad \exists z \forall t \left(\langle x, z \rangle \leq_{\varphi} \langle y, t \rangle\right)$$
  $$x <_{\psi} y \quad \text{iff} \quad \exists z \forall t \left(z = t \lor \langle x, z \rangle <_{\varphi} \langle y, t \rangle\right).$$
The Prewellordering Property

**Question**

We know PWO($\Sigma^1_2$). Does this imply PWO($\Sigma^1_1$) just because $\Sigma^1_1 \subseteq \Sigma^1_2$?

- The answer here is no: every $\Sigma^1_1$-set has a $\Sigma^1_2$-norm, but not necessarily a $\Sigma^1_1$-norm.
- In fact, these three pointclasses ($\Sigma^1_0$, $\Pi^1_1$, and $\Sigma^1_2$) are the only three analytical pointclasses we can show PWO for under ZFC:

\[
\begin{array}{c}
\Sigma^1_0 \\
\Sigma^1_1 \\
\Sigma^1_2 \\
\Sigma^1_3 \\
\Sigma^1_4 \\
\Pi^1_0 \\
\Pi^1_1 \\
\Pi^1_2 \\
\Pi^1_3 \\
\Pi^1_4 \\
\ldots
\end{array}
\]
The Prewellordering Property

- In particular (assuming $\text{Con}(\text{ZFC + PD})$), $\text{PWO}(\Sigma^1_n)$ is independent for odd $n > 2$.
- For even $n > 2$, Harrington has some (unpublished) work which gives a model where $\text{PWO}(\Sigma^1_n)$ and $\text{PWO}(\Pi^1_n)$ both fail.
One question that arises from this picture is can we have both $\text{PWO}(\Sigma^1_n)$ and $\text{PWO}(\Pi^1_n)$? The answer is “no”, and this is a result of properties more topological in nature.

**Theorem (19D • 2, 4, 6, and 7)**

Let $\Gamma$ be an adequate pointclass. Therefore

- $\text{PWO}(\Gamma)$ implies $\Gamma$ has the reduction property.
- $\Gamma$ has the reduction property implies $\neg \Gamma$ has the separation property.
- If $\Gamma$ has a universal set, $\Gamma$ cannot have both the reduction and separation properties.
- Every $\sigma$-algebra (e.g. $\Delta^1_n$) has both the reduction and separation properties.

What these properties 

- What these properties *are* exactly isn’t too important for us.
- They at least establish $\neg (\text{PWO}(\Gamma) \land \text{PWO}(\neg \Gamma))$ for $\Gamma = \Sigma^1_n$.
- They also tell us we shouldn’t be considering $\Delta^1_n$, since they trivially have these properties.
Theorem

The following are equivalent:

1. \( \omega_1^{L[x]} < \omega_1 \).
2. \( \text{PSP}(\Sigma^1_2(x)) \).
3. \( \text{PSP}(\Pi^1_1(x)) \).

- Let’s get back on track.
- The main way we’ll show (3) implies (1) above is to use \( \Pi^1_1 \)-uniformization (aka Kondô’s theorem).
- In general, the way to show \( \Gamma \)-uniformization is with the scale property on \( \Gamma \).
- As usual, we can prove the scale property on \( \Pi^1_1 \) and \( \Sigma^1_2 \), but we can’t go beyond this in ZFC alone.
- Determinacy axioms can push this further with the same alternating pattern as with PWO.
Introducing Scales

Definition

Let $X \subseteq \mathcal{N}$. A scale on $X$ is a sequence $\vec{\varphi} = \langle \varphi_n : n < \omega \rangle$ such that

- each $\varphi_n$ is a norm on $X$;
- for all convergent $x \in \omega \times X$, if each $\varphi_n \circ x$ is eventually constant then
  - $\lim x \in X$ and
  - (lower semi-continuity) $\varphi_n(\lim x) \leq \lim(\varphi_n \circ x)$ for each $n < \omega$.

Without lower semi-continuity, $\vec{\varphi}$ is called a semi-scale.

- As before, we are interested in scales with definability restrictions.
- We can easily get scales on any $X$ we want by using AC.
- $f : X \to |X|$ a bijection with $\varphi_n = f$ for every $n$ works.
- If $x \in \omega \times X$ is convergent and $f \circ x$ is eventually constant then $x$ is eventually constant. So $\lim x \in \text{im } x \subseteq X$.
- Lower semi-continuity is also easy in this case: $f(\lim x) = \lim(f \circ x)$.
- As before, we are interested in scales with definability restrictions.
Introducing Scales

- So what are some less trivial ways to get scales?
- How could scales possibly be useful?

Result (19 E • 3)

For $X \subseteq \mathcal{N}$ and $\kappa \geq \aleph_0$, the following are equivalent.

- $X$ has a scale $\bar{\phi}$ where $\varphi_n(x) < \kappa$ for all $n < \omega$, $x \in X$.
- $X$ has a semi-scale where $\bar{\phi}$ where $\varphi_n(x) < \kappa$ for all $n < \omega$, $x \in X$.
- $X$ is $\kappa$-suslin.

- Suslin representations of sets are very important because they allow us to ask questions about trees instead of whatever weird reals we have.
- We will only prove the downward direction. The upward isn’t very interesting to me.
Result (19 E • 3)

If $X$ has a semi-scale bounded by $\kappa$ then $X$ is $\kappa$-suslin.

Proof.

- A scale is a semi-scale so one direction is obvious.
- If $\vec{\varphi}$ is a semi-scale, consider the tree building up sequences of elements and their norms: $\langle \tau, \rho \rangle \in \omega \times \omega \kappa$ is in $T$ iff there’s an $x \in X$ with
  - $\tau \triangleleft x$;
  - $\rho = \langle \varphi_n(x) : n < \text{lh}(\tau) \rangle$.
- This is a tree over $\omega \times \kappa$. So $p[T]$ is $\kappa$-suslin.
- $\langle x, \langle \varphi_n(x) : n < \omega \rangle \rangle \in [T]$ for any $x \in X$ so $X \subseteq p[T]$. 
Result (19 E • 3)

If $X$ has a semi-scale bounded by $\kappa$ then $X$ is $\kappa$-suslin.

Proof.

- $\langle \tau, \rho \rangle \in T$ iff there's an $x \in X$ with
  - $\tau \triangleleft x$;
  - $\rho = \langle \varphi_n(x) : n < \text{lh}(\tau) \rangle$.
- To show $p[T] \subseteq X$, if $x \in p[T]$, then we get a sequence $\langle x_n \in X : n < \omega \rangle$ witnessing $x \upharpoonright n \in pT$.
- $x \upharpoonright n = x_n \upharpoonright n$ so $\langle x_n \in X : n < \omega \rangle$ converges to $x$ with $\langle \varphi_n(x_m) : m < \omega \rangle$ eventually determined and thus constant.
- Hence $\lim x_n = x \in X$. \hfill $\blacksquare$
• The motivating result for PWO was that WO had a $\Pi^1_1$-norm given by $x \mapsto \|x\|$.
• We get a similar result for scales.

### Result (19 E • 4)

There is a scale on WO. In fact, the relations on triples $\langle x, y, n \rangle \in \mathcal{N}^2 \times \omega$ are both $\Pi^1_1$:

- $x \leq \varphi_n y$ iff $x \in WO \land (y \in WO \rightarrow \varphi_n(x) \leq \varphi_n(y))$
- $x < \varphi_n y$ iff $x \in WO \land (y \in WO \rightarrow \varphi_n(x) \leq \varphi_n(y))$.

• What are $\varphi_n$? They code $\|x\|$ and its initial segment “up to $n$”.
• More precisely, we can consider

$$(E_x)_{<n} = \{\langle a, b \rangle : a \ E x \ b \ E x \ n \neq b\}.$$  

• These $(E_x)_{<n}$ tell us how $E_x$ is built up.
• So we define $\varphi_n(x) = \text{code}(\|E_x\|, \|(E_x)_{<n}\|)$.
• Showing $\langle \varphi_n : n < \omega \rangle$ is in fact a scale is a boring, technical process.
• This motivates the concept of a $\Gamma$-scale.

**Definition**

Let $\Gamma$ be a pointclass. A $\Gamma$-scale is a scale $\bar{\varphi}$ on a set $X$ such that the relations on $\langle x, y, n \rangle \in \mathbb{N}^2 \times \omega$ are in $\Gamma$:

\[
\begin{align*}
ex & \leq \varphi_n y \iff x \in X \wedge (y \in X \rightarrow \varphi_n(x) \leq \varphi_n(y)) \\
ex & < \varphi_n y \iff x \in X \wedge (y \in X \rightarrow \varphi_n(x) \leq \varphi_n(y)).
\end{align*}
\]

• So for adequate pointclasses, $X$ having a $\Gamma$-scale implies $X \in \Gamma$ by $x \in X$ iff $x \leq_{\varphi_0} x$.

• We can also show that every $\Pi^1_1$-set has a $\Pi^1_1$-scale in the same way as with PWO.
Theorem

Every $\Pi^1_1$-set has a $\Pi^1_1$-scale, i.e. the scale property holds for $\Pi^1_1$.

Proof.

- Let $A \in \Pi^1_1$, $A = f^{-1}\text{"WO}$ for some computable $f : \mathcal{N} \to \mathcal{N}$.
- Let $\tilde{\psi}$ a $\Pi^1_1$-scale on WO.
- We can form $\langle \psi_n \circ f : n < \omega \rangle$ and get a $\Pi^1_1$-scale on $A$.

- We can also show the scale property for $\Sigma^1_2$.
- Indeed, we get the same restrictions as with PWO: after $\Sigma^1_2$,
  - the scale property is independent of ZFC;
  - PD gives a full zig-zag pattern; and
  - $V = L$ gives an initial zig followed by a line.
- Again, we mostly care about scales for now to get uniformization.
Definition

Let $X \subseteq A \times B$. A uniformization is a function $f \subseteq X$ such that $\text{dom}(f) = \text{dom}(X)$.

For $\Gamma$ a pointclass, $\Gamma$-uniformization is the statement that every $X \in \Gamma$ has a uniformization $f \in \Gamma$.

Let’s prove $\Pi^1_1$-uniformization now that we have the scale property.

Theorem

$\Pi^1_1$-uniformization holds (and $\boxtimes^1_1$-uniformization).

- Let $A \in \Pi^1_1$ be arbitrary and let $A''x = \{y \in \mathcal{N} : \langle x, y \rangle \in A\}$.
- To get a uniformization, we need to find a unique $y \in A''x$ defined from $x$.
- Uniqueness is the easy part. The hard part is showing existence and ensuring the resulting $f$ is $\Pi^1_1$ with $\text{dom}(f) = \text{dom}(A)$.
- That is where scales come in.
**Theorem**

$\Pi^1_1$-uniformization holds (and $\Pi^1_1$-uniformization).

- Let $x \in A$ be arbitrary and $\bar{\varphi}$ a $\Pi^1_1$-scale on $A \in \Pi^1_1$.
- Set $f(x) = y$ iff $\langle x, y \rangle \in A$ and for all $z \in \mathcal{N}$ and all $n < \omega$,
  - If $(*) \langle x, z \rangle \in A, z \upharpoonright n = y \upharpoonright n$, and $\varphi_m(x, z) = \varphi_m(x, y)$ for all $m < n$,
  - Then $(**)$ $y(n) < z(n)$, or else $y(n) = z(n) \land \varphi_n(x, y) \leq \varphi_n(x, z)$.
- Basically, $y$ is lexicographically least among those whose first norms agree with $y$’s initial first segments.
- Let’s show that this indeed uniquely defines a $y$ (if there exists one).
- Suppose this holds for two $y_1, y_2 \in \mathcal{N}$ and let $N$ be their first disagreement.
- $(*)$ therefore holds inductively for $n \leq N$ (consider $(*)$ with $n = 0$ and then $(**)$ with $n = 0$ to get $\varphi_n(x, y_1) = \varphi_n(x, y_2)$)
- $(**)$ thus holds. But applying $(*)$ with $z = y_1$ and $y = y_2$ gives $y_2(N) < y_1(N)$ and vice versa yields $y_1(N) < y_2(N)$, a contradiction.
Theorem

$\Pi^1_1$-uniformization holds (and $\Sigma^1_1$-uniformization).

- Set $f(x) = y$ iff $\langle x, y \rangle \in A$ and for all $z \in \mathcal{N}$ and all $n < \omega$,
  
  $\text{If } (*) \langle x, z \rangle \in A, z \upharpoonright n = y \upharpoonright n, \text{ and } \varphi_m(x, z) = \varphi_m(x, y) \text{ for all } m < n,
  
  \text{Then } (**) y(n) < z(n), \text{ or else } y(n) = z(n) \land \varphi_n(x, y) \leq \varphi_n(x, z)$.

- It’s not hard to see that $(*)$ is $\Sigma^1_1$ and $(**)$ is $\Pi^1_1$.

- Thus $f$ (if well-defined) will be $\forall \mathcal{N} (\neg \Sigma^1_1 \lor \Pi^1_1) = \Pi^1_1$.

- So we need to show there is such a $y$, which is much harder.

- We use the scale to do the heavy lifting in the construction and ensure the limit (i.e. end result) is in $A$.

- Start out with $A_0 = A''x$.

- At stage $n + 1$, we first consider the set of all $y \in A_n$ with minimal $\varphi_n(x, y)$ (among other elements in $A_n$).

- Then we thin out that set to only consider the $y$s with minimal $y(n)$. The result is $A_{n+1}$.

- We now want a $y \in \bigcap_{n<\omega} A_n$.
Theorem

$\Pi^1_1$-uniformization holds (and $\Sigma^1_1$-uniformization).

- Start out with $A_0 = A''x$.
- At stage $n + 1$, we first consider the set of all $y \in A_n$ with minimal $\varphi_n(x, y)$ (among other elements in $A_n$).
- Then we thin out that set to only consider the $y$s with minimal $y(n)$. The result is $A_{n+1}$.
- Any sequence $\langle y_n \in A_n : n < \omega \rangle$ is necessarily convergent since we’re deciding more and more when moving from $A_n$ to $A_{n+1}$.
- Such a sequence also has $\langle \varphi_n(y_k) : k < \omega \rangle$ as eventually constant because again we’re deciding more of the norm moving from $A_n$ to $A_{n+1}$.
- As a scale, it follows that $y = \lim_{n \to \infty} y_n \in A$ and an easy induction shows it satisfies the definition of $f(x) = y$.
- Hence $f$ is well-defined and a $\Pi^1_1$-uniformization of $A$.  \[ \]
• Again, we have the same picture with uniformization as with PWO, scales, and other properties: $\Sigma^1_2$ is the limit of what can be shown in ZFC.

• The zig-zag (or lackthereof in L) of PD also holds with uniformization.

• This is quite useful with determinacy because uniformization is a choice-like principle telling us not only that we can pick out an element, but we can do so in a simply definable way.

• Let’s return to the main goal here:

**Theorem**

The following are equivalent:

1. For every $x \in \mathcal{N}$, $L[x] \models \omega^V_1$ is inaccessible’.
2. $\text{PSP}(\Pi_1^1)$. 
What was all this about again?

- Restated, the main goal is the following.

**Theorem**

\[
\text{PSP}(\Pi^1_1(x)) \text{ implies } \omega^L_1[x] < \omega_1 \text{ for every } x \in \mathcal{N}.
\]

**Proof.**

- Let \( x \in \mathcal{N} \) and suppose \( \omega^L_1[X] = \omega_1 \), aiming to show \( \neg \text{PSP}(\Pi^1_1(x)) \).
- For each \( \alpha < \omega^L_1[x] \), let \( f(\alpha) \) be the \( \prec_{L[x]} \)-least real in \( \text{WO} \cap L[x] \) coding \( \langle \alpha, \in \rangle \).
- Set \( X = f^"\omega^L_1[x] \) so that every ordinal is coded by an element of \( X \).
- We actually considered this set already to show that \( L \models \neg \text{PSP}(\Sigma^1_2) \).
- The relevant theorems generalize to show
  1. \( X \in \Sigma^1_2(x) \) is uncountable (since \( \aleph^L_1[x] = \aleph_1 \)), and
  2. \( X \) has no uncountable \( \Sigma^1_1 \)-subset (by the Boundedness lemma).
What was all this about again?

**Theorem**

\[ \text{PSP}(\Pi^1_1(x)) \text{ implies } \omega_1^{L[x]} < \omega_1 \text{ for every } x \in \mathcal{N}. \]

**Proof.**

- \( X = f'' \omega_1^{L[x]} \) where \( f(\alpha) \in \text{WO} \cap L[x] \) codes \( \alpha \).
  1. \( X \in \Sigma^1_2(x) \) is uncountable (since \( \aleph_1^{L[x]} = \aleph_1 \)), and
  2. \( X \) has no uncountable \( \Sigma^1_1 \)-subset (by the Boundedness lemma).
- Let \( X = pY \) for \( Y \in \Pi^1_1(x) \).
- By \( \Pi^1_1(x) \)-uniformization, we get a \( \Pi^1_1 \)-function \( f \subseteq Y \) with \( \text{dom}(f) = X \).
- Hence \( |f| = |X| = \aleph_1 \).
- \( f \) also can’t have any perfect subset. To see this, firstly any perfect subset must be closed and uncountable.
- But any closed \( g \subseteq f \) has \( \text{dom}(g) \in \Sigma^1_1 \) and is thus countable since \( \text{dom}(g) \subseteq X \) and (2) holds.
- Thus \( \neg \text{PSP}(\Pi^1_1(x)) \).
• Leo Harrington, The constructible reals can be (almost) anything, 1974. Unpublished.


